VII. *On Centripetal Forces.* *By* Edward Waring, *M. D. F. R. S. Profeſſor of Mathematics at* Cambridge.

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PROP. I.

1. LET a curve PpN (Tab. II. fig. 1.), of which the perpendiculars to the two neareſt points P and p of the curve are PO and pO, and conſequently O the center of a circle, which has the ſame curvature as the given curve in the point P; draw PY and ly tangents to the curve in the points P and p; from S draw Sy and ShY reſpectively perpendiculars to the tangents ly and PY; and let ShY cut the tangent ly in h; then will ultimately hY $(-\dot{P})$ be the decrement of the perpendicular $SY = P$; and the triangles lhY and POp be ſimilar: for the angles POp and hlY are equal, and the angles lYh and OPp right ones; therefore $PO : Pp :: lY$ ultimately $= PY : Yh$ decrement of the perpendicular, whence $Pp = \frac{Yh \times PO}{lY} = \frac{Yh \times PO}{PY}$.

1.2. Fig. 2. and 1. The force in the direction PS is as the ultimate ratio of $2 \times QR$ (the ſpace through which a body is drawn from the direction of its motion in the tangent in a given time towards the center of force); but ultimately $2QR = \frac{2QP^2}{PV}$, where QP is as the ſpace deſcribed in a given time, and conſequently as the velocity (V) of the body at the given point P, and PV the chord of curvature in the direction SP.

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1.3. The increment (Pp) of the ſpace divided by the velocity V is ultimately as the increment of the time, and $=$ the increment of the velocity ($\dot{V}$) divided by the force $\frac{2V^2}{PV} \times \frac{PY}{SP}$ in the direction of the tangent, that is, $\frac{Pp}{V} = \frac{\dot{V} \times PV \times SP}{2V^2 \times PY}$; for Pp ſubſtitute $\frac{Yh \times PO}{PY}$, and there reſults $\frac{-\dot{P} \times PO}{PY \times V} = \frac{\dot{V} \times PV \times SP}{2V^2 \times PY}$; and conſequently $\frac{-2\dot{P} \times PO}{SP \times PV} = \frac{\dot{V}}{V}$; but $\frac{SP \times PV}{2PO} = SY = P$, whence $\frac{-\dot{P}}{P} = \frac{\dot{V}}{V}$, and $V = \frac{a}{P}$, where a is an invariable quantity.

Cor. Since $V \times P$, that is, SY the perpendicular multiplied into the velocity (which is ultimately as Pp the ſpace deſcribed in a given time) is ultimately as the area deſcribed round the center S in a given time; but this rectangle $= a$, a given quantity; therefore the area, deſcribed round the center of force S in a given time, will be a given quantity, and thence in unequal times will be proportional to the times.

1.4. The ſagitta QR is ultimately as the force, when the time is given; and when the time is not given, it will be as the force into the ſquare of the time; from which expreſſion, by ſubſtituting for QR and the time their values, may be deduced ſeveral others.

Sir ISAAC NEWTON has demonſtrated this propoſition with the greateſt ſimplicity; and this is given to ſhew, that the ſame propoſition may be deduced from different principles.

PROP. II.

1. Fig. 3. Given the relation between SP′ the diſtance from a point S, and SY′ a perpendicular from the point S to P′Y, a line touching a curve in the point P′; to find the relation

between Sp' and Sy' (a perpendicular from the point S to $p'y$, a line touching the curve in the point p'); in which two curves $PP'L$ and $pp'l$, the forces and velocities at any equal diſtances SP and Sp are equal, and conſequently the perpendiculars SY and Sy, at the above-mentioned equal diſtances SP and Sp are to each other in a given ratio N : n.

In the equation expreſſing the relation between SP' and SY' for SP' and SY' write reſpectively Sp and $\frac{Sy' \times N}{n}$, and there reſults the equation ſought: for the diſtances SP and Sp' being equal, the perpendiculars SY' and Sy' are as N : n.

Ex. 1. Let S be the focus of a conic ſection, then will $\frac{1}{4} C^2 \times \frac{D}{T \pm D} = SY^2 = P^2$, where T and C denote its tranſverſe and conjugate axes, and D the diſtance SP; for P write $\frac{N}{n} \times p$, and there reſults the equation $\frac{1}{4} C^2 \times \frac{D}{T \pm D} = \frac{N^2}{n^2} \times p^2$, which is an equation to a conic ſection of the ſame name (*viz.* ellipſe, parabola, or hyperbola) as the given curve, of which the tranſverſe axis is T, and conjugate $= \frac{C \times n}{N}$, and perpendicular from the focus to the tangent $= p$. If T and C are infinite, and conſequently the curve a parabola, and the equation $\frac{1}{4} L \times D = P^2$, then will the *latus rectum* of the reſulting equation be $\frac{L \times n^2}{N^2}$.

Ex. 2. Let S be the center of the logarithmic ſpiral, then will the equation be $a \times SP = a \times D = SY = P$, and conſequently the reſulting equation $a \times D = \frac{N}{n} \times p$, whence $\frac{an}{N} \times D = p$ an equation to a logarithmic ſpiral having the ſame center.

Ex. 3. Let T and C be the ſemi-conjugate axes of a conic ſection, and S its center; then will the equation expreſſing the relation between the diſtance D and perpendicular P be $D^2 \pm \frac{T^2C^2}{P^2} = T^2 \pm C^2$; for P write as before $\frac{Np}{n}$, and there reſults the equation $D^2 \pm \frac{n^2T^2C^2}{N^2p^2} = T^2 \pm C^2$, an equation to a conic ſection of the ſame name, of which the tranſverſe and conjugate diameters are reſpectively two roots (x) of the equation $x^2 \pm \frac{n^2T^2C^2}{N^2x^2} = T^2 \pm C^2$, becauſe in this caſe $p = D$.

The ſum or difference of the ſquares of the tranſverſe and conjugate diameters, in all the reſulting equations, will be the ſame.

Cor. In every equal diſtance, the chord of curvature paſſing through the center of force is the ſame; for the forces in that direction, and the velocities at every equal altitude are the ſame.

PROP. III.

1. Fig. 4. and 3. Given an equation $A = o$, expreſſing the relation between the abſciſs $SM = x$ and ordinate $MP = y$; to find the equation expreſſing the relation between $SP = \sqrt{x^2 + y^2}$ and $SY = P$, the perpendicular from S on the tangent PY. From the equation $A = o$ find $\dot{x} = B\dot{y}$, which ſubſtitute for $\dot{x}$ in the equation $(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \times P = \dot{x}y \pm x\dot{y}$ deduced from the ſimilar triangles Plo, MTP, and STY, where $lo = \dot{x}$ and $Po = \dot{y}$; let the reſulting equation be $C = o$; reduce the three equations $A = o$, $C = o$, and $x^2 + y^2 = SP^2 = D^2$ into one, ſo that the unknown quantities x and y may be exterminated, and there reſults an equation expreſſing the relation between D and P.

Cor. Hence from the equation expreſſing the relation between x and y, the abſciſs and ordinate of a curve, can be deduced an equation expreſſing the relation between the diſtance SP and perpendicular SY; and from the equation expreſſing the relation between the diſtance SP and SY can be deduced an equation expreſſing the relation between the diſtance Sp and perpendicular Sy from the point S to the tangent py of a curve, whoſe force and velocity at every equal diſtance is the ſame as in the given curve, but the direction different.

2. Given an equation $K=o$ expreſſing the relation between $SP=D$ and $SY=P$; to find an equation expreſſing the relation between $SM=x$ and $PM=y$, the abſciſs and ordinate of the ſame curve.

In the given equation $K=o$ for D and P write reſpectively $\sqrt{x^2+y^2}$ and $\frac{y\dot{x}\pm x\dot{y}}{\sqrt{\dot{y}^2+\dot{x}^2}}$, and there reſults a fluxional equation $L=o$ of the firſt order, of which the fluent expreſſes the general relation between x and y.

Cor. If in the given equation for P be wrote nP', there reſults the equation $K'=o$, which expreſſes the relation between the perpendicular $Sy=P'$ and diſtance $Sp=D'$ of every curve, which at equal diſtances has the ſame velocity and force tending to S; reduce the equations $K'=o$, $D=\sqrt{x^2+y^2}$ and $nP'=\frac{y\dot{x}\pm x\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ into one, ſo that D and P′ may be exterminated, and there will reſult the ſame fluxional equation of the firſt order, expreſſing the relation between x, y, and their fluxions, whatever may be the value of n. The general fluent of this fluxional equation contains the relation between the abſciſs and ordinates of all curves, which have the ſame force and velocity at the ſame diſtance as the force and velocity in the given curve.

PROP. IV.

1. Let a body move in a given curve PH (fig. 5.), of which the velocity (v) at any point P is given: and let the forces f'', f''', &c. tending to all the given centers S'', S''', &c. (except two S and S') be given; to find the forces f and f' tending to the two points S and S'.

Draw a line PO perpendicular to the tangent yPy'; and from the given centers S, S', S'', &c. draw lines Sl and Sy, $S'l'$ and $S'y'$, $S''l''$ and $S''y''$, &c. perpendicular to the lines PO and yPy', &c.; then will $\frac{v^2}{PO} = f \times \frac{Pl}{PS} \pm f' \times \frac{Pl'}{PS'} \pm f'' \times \frac{Pl''}{PS''} \pm$ &c. (where PO is the radius of the circle having the ſame curvature as the curve in the point P), and $\frac{-v\dot{v}}{\dot{A}} = f \times \frac{Pv}{PS} \pm f \times \frac{Py'}{PS} \pm$ &c. (where A denotes the arc of the curve PH); from the *data* may be deduced all the quantities contained in the above mentioned two equations, except f and f'; and conſequently from the two given ſimple equations be deduced the forces ſought f and f'.

2. Let the velocity of the body moving in the given curve be ſuppoſed always uniform; then $f \times \frac{Py}{PS} \pm f' \times \frac{Py'}{PS'} \pm f'' \times \frac{Py'}{PS''} \pm$ &c. $= 0$.

Ex. Let the curve HPl be an ellipſe, and the two foci S and S' the centers of forces; then will $f \times \frac{Py}{SP} = f' \times \frac{Py'}{SP'}$, but the angle $SPy = S'Py'$, and conſequently $\frac{Py}{SP} = \frac{Py'}{S'P}$ and $f = f'$; but ſince $\frac{v^2}{PO} = f \times \frac{Sy}{SP} + f \times \frac{Sy'}{SP} = 2f \times \frac{Sy}{SP}$, and $v = a$, then will $f = \frac{a^2 \times SP}{2Sy \times PO}$ be the force tending to each focus.

In theſe and the ſubſequent caſes the lines Py, Py', Py'', &c. are to be taken negatively or affirmatively, as they are ſituated on the ſame or different ſides of P; and in the ſame manner the lines Pl, Pl', Pl'', &c. are to be taken negatively or affirmatively as they are ſituated on the ſame or different ſides of the tangent yPy', &c.

3. Let the centers M, M$'$, M$''$, M$'''$, &c. of forces be points not ſituated in the plane of the given curve HPI, &c. and the forces f''', f'''', &c. tending to each of the centers M$'''$, M$''''$, &c. (except three M, M$'$, and M$''$) be given; to find the forces f, f', and f'' tending to thoſe three points M, M$'$, and M$''$.

Draw MS, M$'$S$'$, M$''$S$''$, &c. perpendicular to the plane HPI, &c. from the above-mentioned points, and aſſume the equation $f \times \frac{\mathrm{MS}}{\sqrt{\mathrm{MS}^2 + \mathrm{SP}^2}} \pm f' \times \frac{\mathrm{M'S'}}{\sqrt{\mathrm{M'S'}^2 + \mathrm{S'P}^2}} \pm f'' \times \frac{\mathrm{M''S''}}{\sqrt{\mathrm{M''S''}^2 + \mathrm{S''P}^2}} \pm f''' \times \frac{\mathrm{M'''S'''}}{\sqrt{\mathrm{M'''S'''}^2 + \mathrm{S'''P}^2}} + \&c. = 0$, and the two preceding equations $\frac{v^2}{\mathrm{PO}} = f \times \frac{\mathrm{P}l}{\mathrm{PM}} \pm f' \times \frac{\mathrm{P}l'}{\mathrm{PM'}} \pm f'' \times \frac{\mathrm{P}l''}{\mathrm{PM''}} \pm$ &c. and $\frac{-v\dot{v}}{\dot{\mathrm{A}}} = f \times \frac{\mathrm{P}y}{\mathrm{PM}} \pm f' \times \frac{\mathrm{P}y'}{\mathrm{PM'}} \pm f'' \times \frac{\mathrm{P}y''}{\mathrm{PM''}} \pm$ &c.; from the *data* may be found all the quantities f''', f'''', &c.; and conſequently from the above mentioned equations may be deduced the forces f, f', and f''.

4. Let the body move in different planes, that is, in a curve of double curvature at the ſame points; draw PR a tangent to the curve at the point P, and PQ an arc of the curve of double curvature; draw alſo two planes PRV and PRT, cutting one another in the line PR; from the point Q let fall QV and QT perpendicular to thoſe planes reſpectively, and from the points V and T draw Vv and Tt reſpectively perpendicular to the line PR; let v be the velocity of a body moving in the given

curve at the point P, and aſſume $\frac{PQ^2}{Vv} = 2C$ and $\frac{PQ^2}{Tt} = 2C'$ reſpectively; from the given centers of forces M, M′, M″, M‴, &c. draw MS, MS′, M″S″, M‴S‴, &c.; Ms, M′s', M″s'', M‴s''', &c. reſpectively perpendicular to the two planes RPV and RPT; and PL and Pl perpendicular to the line PR in the ſame two planes RPV and RPT; and alſo SP, S′P, S″P, S‴P, &c.; sP, s'P, s''P, &c.: from the points S, S′, S″, S‴, &c. s, s', s'', &c. draw the lines SH, S′H′, S″H″, S‴H‴, &c. sh, $s'h'$, $s''h''$, $s'''h'''$, &c. reſpectively perpendicular to the lines PL and Pl; and SK, S′K′, S″K″, S‴K‴, &c. sk, $s'k'$, $s''k''$, $s'''k'''$, &c. perpendicular to the line RP; and let the forces f''', f'''', &c. tending to all the points M‴, M⁗, &c. (except three, M, M′, and M″) be given; then from the three given equations $\frac{v^2}{C} = \frac{PH}{MP} \times f \pm \frac{PH'}{M'P} \times f' \pm \frac{PH''}{M''P} \times f'' \pm$ &c. and $\frac{v^2}{C'} = \frac{Ph}{MP} \times f \pm \frac{Ph'}{M'P} \times f' \pm \frac{Ph''}{M''P} \times f'' \pm$&c. and $\frac{-v\dot{v}}{\dot{A}} = \frac{PK}{MP} \times f \pm \frac{PK'}{M'P} \times f' \pm \frac{PK''}{M''P} \times f'' \pm \frac{PK'''}{M'''P} \times f''' \pm$ &c. $= \frac{Pk}{MP} \times f \pm \frac{Pk'}{M'P} \times f \pm \frac{Pk''}{M''P} \times f''$ $\pm$ &c. which contain only three unknown quantities, can be deduced the forces f, f', and f'', required, tending to the points M, M′, and M″.

PROP. V.

Let a body acted on by forces tending to any given points S, S′, S″, &c. move in a given curve, to find its velocity in any point of the curve.

Find the fluent of the fluxion ($f \times \frac{Py}{PS} \pm f' \times \frac{Py'}{PS'} \pm f'' \times \frac{Py''}{PS''} \pm f''' \times \frac{Py'''}{PS'''} \pm$ &c.) $\dot{A} = f\dot{D} \pm f'\dot{D}' \pm f''\dot{D}'' \pm$ &c. $= - v\dot{v}$ when the forces are all contained in the ſame plane; or the fluent of

$(f \times \frac{Py}{PM} \pm f' \times \frac{Py'}{PM'} \pm f'' \times \frac{Py''}{PM''} \pm \&c.) \times \dot{A}$ (when contained in different planes) $= f \times \dot{P}M \pm f' \times \dot{P}M' \pm f'' \times \dot{P}M'' \pm \&c. = f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm \&c.$; but ſince f, f', f'', &c. are given functions of the quantities D, D′, D″, &c. the fluents of $f \times \dot{D}$, $f' \times \dot{D}'$, $f'' \times \dot{D}''$, &c. can be found; which, when properly corrected will be as $\frac{v^2}{2} = \frac{1}{2}$ the ſquare of the velocity in any point P. A denotes the arc of the curve, and D, D′, D″, &c. the reſpective diſtances of the body from the centers of forces.

Cor. The increment of the time of deſcribing any arc of the above-mentioned curve will be as the increment of the arc $= \dot{A}$ divided by the velocity found above, and conſequently the time itſelf will be as the fluent of it properly corrected.

PROP. VI.

1. Let a body move in any curve, and be acted on by forces tending to any given points, S, S′, S″, S‴, &c.; all of which, except the force f tending to the point S, let be given, to find f the force tending to S.

Let Sy, S′y', S″y'', &c. be perpendicular to the tangent Py of the curve at the point P; reſolve the forces f, f', f'', f''', &c. tending to S, S′, S″, S‴, &c. reſpectively into two forces, of which one acts perpendicular to Py, the other, Sl, S′l', S″l'', &c. perpendicular to PO, which is perpendicular to Py; let PO be radius of the circle of the ſame curvature as the curve, and v the velocity of the body at the point P; then will $\frac{v^2}{PO} = f \times \frac{Sy}{SP} \pm f' \times \frac{S'y'}{S'P} \pm f'' \times \frac{S''y''}{S''P} \pm f''' \times \frac{S'''P}{S'''y'''} \pm \&c.$ and $-\frac{v\dot{v}}{\dot{A}} = f \times \frac{Py}{SP} \pm f' \times \frac{Py'}{S'P} \pm f'' \times \frac{Py''}{S''P} \pm f''' \times \frac{Py'''}{S'''P} \pm \&c.$: for $\frac{Sy \times PO}{SP} =$

$\frac{1}{2}$ chord of the circle of curvature, which paſſes through S, write C; and for $\mathrm{PO} \times (\pm f' \times \frac{\mathrm{S}'y'}{\mathrm{S}'\mathrm{P}} \pm f'' \times \frac{\mathrm{S}''y''}{\mathrm{S}''\mathrm{P}} \pm \&c.)$ ſubſtitute H, and for $\frac{\mathrm{P}y}{\mathrm{SP}}$ write B; and for $\pm f' \times \frac{\mathrm{P}y'}{\mathrm{S}'\mathrm{P}} \pm f'' \times \frac{\mathrm{P}y''}{\mathrm{S}''\mathrm{P}} + \&c.$ ſubſtitute D, and the two preceding equations become $v^2 = f \times \mathrm{C} + \mathrm{H}$ and $-v\dot{v} = (\mathrm{B}f + \mathrm{D})\,\dot{\mathrm{A}}$, where $\dot{\mathrm{A}}$ denotes as before the increment of the arc of the curve ! from the firſt equation $v\dot{v} = \frac{f \times \dot{\mathrm{C}} + \mathrm{C}\dot{f} + \dot{\mathrm{H}}}{2} = -(\mathrm{B}f\dot{\mathrm{A}} + \mathrm{D}\dot{\mathrm{A}})$ and conſequently $\mathrm{C}\dot{f} + (\dot{\mathrm{C}} + 2\mathrm{B}\dot{\mathrm{A}})f + \dot{\mathrm{H}} + 2\mathrm{D}\dot{\mathrm{A}} = o$, from which fluxional equation may be deduced the force f tending to the center (S) $= -\mathrm{C}^{-1} \times e^{-\int \frac{2\mathrm{B}\dot{\mathrm{A}}}{\mathrm{C}}} \times \int (\dot{\mathrm{H}} + 2\mathrm{D}\dot{\mathrm{A}}) \times e^{\int \frac{2\mathrm{B}\dot{\mathrm{A}}}{\mathrm{C}}}$, where e is the number, whoſe hyper. log. = 1.

Cor. Fig. 1. Let f', f'', f''', &c. be each $=o$, then will $\mathrm{D} = o$, $\mathrm{H} = o$; and conſequently $\int (2\dot{\mathrm{D}}\mathrm{A} + \dot{\mathrm{H}}) \times e^{\int \frac{2\mathrm{B}\dot{\mathrm{A}}}{\mathrm{C}}} =$ conſt $= a$, and $f = -a \times \mathrm{C}^{-1} \times e^{-\int \frac{2\mathrm{B}\dot{\mathrm{A}}}{\mathrm{C}} = \frac{2\mathrm{B} \times \mathrm{S}\dot{y} \times \mathrm{PO}}{\mathrm{C} \times \mathrm{P}y} = \frac{2\mathrm{S}\dot{y}}{\mathrm{S}y}} = -a\mathrm{C}^{-1} \times e^{-\int \frac{2\mathrm{S}\dot{y}}{\mathrm{S}y}}$; whence $f = \frac{-a}{\mathrm{S}y^2 \times \mathrm{C}}$ as is generally known, where a denotes an invariable quantity.

Cor. The force f being found, the ſquare of the velocity may be deduced from the equation $v^2 = f \times \mathrm{C} + \mathrm{H}$, and the time from the fluent of the fluxion $\frac{\dot{\mathrm{A}}}{v} = \frac{\dot{\mathrm{A}}}{\sqrt{f \times \mathrm{C} + \mathrm{H}}}$.

2. Let the body move in a curve of double curvature, and let the forces f'', f''', &c. tending to all the points M″, M‴, &c. (except two, M and M′) be given; to find the forces tending to the points M and M′.

Aſſume the three equations before given in Prop. 4. *viz.* $\frac{v^2}{C}=\frac{PH}{MP}\times f\pm\frac{PH'}{M'P}\times f'\pm\frac{PH''}{M''P}\times f''\pm$ &c. $\frac{v^2}{C'}=\frac{Pb}{MP}\times f\pm\frac{Pb'}{M'P}\times f\pm\frac{Pb''}{M''P}\times f''\pm$&c. and $-v\dot{v}=\left(\frac{PK}{MP}\times f\pm\frac{PK'}{M'P}\times f'\pm\frac{PK''}{M''P}\times f''\pm\right.$ &c. $=\frac{Pk}{MP}\times f\pm\frac{Pk'}{M'P}\times f'\pm$&c.$\left.\right)\times\dot{A}$, from the two former may be deduced the equations $v\dot{v}=\alpha\dot{f}+\beta\dot{f}'+f\dot{\alpha}+f'\dot{\beta}+\dot{\gamma}$, and $v\dot{v}=\alpha'\dot{f}+\beta'\dot{f}'+f\dot{\alpha}+f'\dot{\beta}'+\dot{\gamma}'$, where $\alpha=\frac{C\times PH}{2MP}$, $\beta=\pm\frac{C\times PH'}{2M'P}$, $\gamma=\pm\frac{1}{2}C\left(\frac{PH''\times f''}{M''P}\pm\frac{PH'''\times f'''}{M'''P}\pm\right.$ &c.$\left.\right)$; $\alpha'=\frac{C'\times Pb}{2MP}$, $\beta'=\pm\frac{C'\times Pb'}{2M'P}$, $\gamma'=\pm\frac{1}{2}C'\left(\frac{Pb''\times f''}{M''P''}\pm\frac{Pb'''\times f'''}{M'''P}\pm\right.$&c.$\left.\right)$, whence may be derived the two equations $\alpha\dot{f}+\beta\dot{f}'+f\dot{\alpha}+f'\dot{\beta}+\dot{\gamma}=\alpha'\dot{f}+\dot{f}'+f\dot{\alpha}'+f'\dot{\beta}'+\dot{\gamma}'=\pi f\pm\rho f'\pm\sigma$, where $\pi=-\left(\frac{PK}{MP}=\frac{Pk}{MP}\right)\times\dot{A}$, $\rho=\pm\left(\frac{PK'}{M'P}=\frac{Pk'}{M'P}\right)\times\dot{A}$, $\sigma=-\left(\pm\frac{PK''}{M''P}\times f''\pm\frac{PK'''}{M'''P}\times f'''\pm\right.$&c. $=$ &c.$\left.\right)\times\dot{A}$.

Reduce theſe two equations to one, ſo that f', f'', &c. and their fluxions, may be exterminated, and there reſults a fluxional equation of the formula $H\ddot{f}+K\dot{f}+Lf+M=o$, where H, K, L, and M, are functions of one of the before mentioned variable quantities (for example, $MP=W$) which may be ſuppoſed to flow uniformly, and its fluxion.

PROP. VII.

1. Fig. 6. Given the force tending to any point S, the velocity and direction of the body; to find the curve deſcribed.

Let the body acted on by a force f tending to S, at the diſtance D′ from S be projected in the direction P′Y′, with a velocity H; and let the perpendicular from S to the tangent P′Y′ be A; from the general fluent of $f\times\dot{D}$, where D denotes the diſtance from S, and f is a function of D, properly corrected find its velocity V at diſtance D, and conſequently the perpendicular SY from the center S to the tangent PY at diſtance D = SP, which will be $\frac{A\times H}{V}=SY$; but A and H are given quantities, and V a known function of D; therefore SY and $\sqrt{SP^2(D^2)-SY^2}=PY$ will be known functions of D; and from the ſimilar triangles SPY and PQT may be deduced $PY : SY :: PT=\dot{D} : QT$, and conſequently $SP\times QT=D\times\frac{\dot{D}\times SY}{PY}$ (which is a known function of D multiplied into $\dot{D}$) will be as the increment of the area deſcribed round the center of force, of which the fluent properly corrected is proportional to the area deſcribed round the center of force, and conſequently to the time. In like manner, $\frac{Sy\times\dot{D}}{D\times PY}=\frac{QT}{SP}$ (proportional to the increment of the angle deſcribed by the body round S) is a function of D multiplied into $\dot{D}$, of which the fluent properly corrected, or angle, will be as a function of D.

1.2. Fig. 7. Given the above-mentioned force, &c.; to find an equation expreſſing the relation between the abſciſs $SM=x$ and ordinate $MP=y$ of the curve deſcribed, and their fluxions.

From the ſimilar triangles P*po* and LPM can be deduced $po=\dot{y} : oP=\dot{x} :: PM=y : LM=\frac{y\dot{x}}{\dot{y}}$; but $LM\pm SM=\frac{y\dot{x}}{\dot{y}}\pm x=\frac{y\dot{x}\pm x\dot{y}}{\dot{y}}=LS$; and conſeqnently $Pp=\sqrt{\dot{x}^2+\dot{y}^2} : po=\dot{y} :: LS=\frac{y\dot{x}\pm x\dot{y}}{\dot{y}} : SY=\frac{y\dot{x}\pm x\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$; but SY is a function to be deduced as above of $SP=\sqrt{x^2+y^2}$, whence the fluxional equation $\frac{y\dot{x}\pm x\dot{y}}{\sqrt{x^2+y^2}}=\varphi:(x^2+y^2)$.

2. Fig. 8. Let a body be acted on by any number of forces (f, f', f'', f''', &c.) in the ſame plane tending to the given points S, S′, S″, S‴, &c.; to find an equation expreſſing the relation between $SP=D$ and $S'P=D'$, and their fluxions, where P is a point ſituated in the curve which the body deſcribes.

Suppoſe YP a tangent to the curve at the point P, and PZ perpendicular to it; and reſolve all the forces tending to S, S′, S″, &c. reſpectively into two others; one in the direction PY, and the other in the direction PZ; ſubſtitute for SP, S′P, S″P, S‴P, &c. reſpectively D, D′, D″, D‴, &c.; and ſuppoſe SY, S′Y′, S″Y″, S‴Y‴, &c. perpendicular to the line PY; then will the triangles PQT and SPY, PQ′T′ and S′PY′ be ſimilar, where PQ denotes a very ſmall arc, and QT and QT′ are perpendicular to the lines SP and S′P; hence $PQ=\frac{PT\times SP}{PY}=\frac{\dot{D}\times D}{PY}=\frac{PT'\times S'P}{PY'}=\frac{\dot{D}'\times D'}{PY'}$; and conſequently $PY:PY'::\dot{D}\times D:\dot{D}'\times D'$; and if the quantities D, D′, $\dot{D}$ and $\dot{D}'$ are given, the ratio of $PY:PY'$ will be given; which being given, together with the line $SS'=a$, the lines PY and PY′, SY and S′Y′, can be found; for, drawing SL parallel to PY, and meeting S′Y′ in L, let $PY'=m\times PY$, then $YY'=(m\pm 1)\,PY=SL$, $SY=\sqrt{(SP^2-PY^2)}=\sqrt{(D^2-PY^2)}$, $S'Y'=\sqrt{(S'P^2-PY'^2)}=(D'^2-m^2\times PY^2)$, $LS'=S'Y'\pm SY=\pm\sqrt{(D'^2-m^2PY^2)}\pm\sqrt{(D^2-PY^2)}$; and $SS'^2=SL^2+LS'^2$ an equation in which all quantities (except PY) are given, and conſequently PY is determined by an equation, which will be a quadratic; but PY being found, from thence PY′, SY and

$S'Y'$ may be deduced, which are conſequently all functions of D, D', $\dot{D}$, $\dot{D}'$, and invariable quantities; and their fluxions $P\dot{Y}'$, $S\dot{Y}$, and $S'\dot{Y}'$ functions of D, D', $\dot{D}$ $\dot{D}'$, $\ddot{D}$, and $\ddot{D}'$: from the ſimilar triangles before given $S\dot{Y} : PQ = \frac{D\dot{D}}{PY} :: PY : \frac{D\dot{D}}{S\dot{Y}} = PO$ the radius of curvature hence PO is a function of D, D', $\dot{D}$, $\dot{D}'$, and $\ddot{D}'$, if $\ddot{D} = o$; and from D, D', SS', $\dot{D}$, $\dot{D}'$, and the point S'' given in poſition can be determined $S''P$, $S''Y''$ and PY''; for let $S''b = C$ be drawn perpendicular to $SS' = a$, and $Sb = b$; then will Sl (if Pl be a perpendicular from the point P to the line SS') $= \pm \frac{a^2 + D^2 - D'^2}{2a}$, and $S'l = \pm \frac{a^2 + D'^2 - D^2}{2a}$, and $Pl = \sqrt{(SP^2 - Sl^2)}$, and $S''P = \sqrt{((b \pm Sl)^2 + (C \pm Pl)^2)}$; draw $S''Y''$ perpendicular to the tangent PY, and cutting the lines SS' and SK parallel to PY in o and n reſpectively; then will $ob = \frac{C \times \sqrt{(SS'^2 - (PY \pm PY')^2)}}{(PY \pm PY') = l}$, $S''o = \frac{C \times bS'}{l}$; (and from the ſimilar triangles $S''ob$ and Son) $on = (b \pm ob) \times \frac{ob}{S''o}$; whence $S''Y'' = \pm S''o \pm on \pm SY$ will be a known function of D, D', $\dot{D}$ and $\dot{D}'$, and invariable quantities: the ſame may be predicated of ſimilar lines drawn to the centers S''', S'''', &c.; and conſequently $(f \times \frac{P\dot{Y}}{SP} \pm f' \times \frac{PY'}{S'P} \pm f'' \times \frac{PY''}{S''P} \pm f''' \times \frac{PY'''}{S'''P} \pm \&c.) \times \dot{A}$ (where $\dot{A}$, as before, denotes the fluxion of the arc of the curve) $= f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm f''' \times \dot{D}''' \pm \&c. = -v\dot{v}$, if v denotes the velocity; but as f, f', f'', f''', &c. are functions of D, D', D'', D''', &c. reſpectively, the fluent of the above mentioned quantity $f\dot{D} \pm f'\dot{D}' \pm f''\dot{D}'' \pm \&c.$ can be found in terms of D, D',

D'',

D'', D''', &c. from the fluents of the fluxions $f\dot{D}$, $f'\dot{D}'$, &c.; and conſequently in terms of D and D′, which let be Z, then will $Z = \frac{-v^2}{2}$; but $v^2 = -2Z = PO \times (f \times \frac{SY}{SP} \pm f' \times \frac{S'Y'}{S'P} \pm f'' \times \frac{S''Y''}{S''P} \pm f''' \times \frac{S'''Y'''}{S'''P} \pm \&c.)$ a fluxional equation of the ſecond order expreſſing the relation between D and D′, and their fluxions.

2. To find an equation expreſſing the relation between $x = SM$ and $y = MP$, where SM (x) is the abſciſs beginning from S and continued in the line SS′, and MP (y) the perpendicular ordinate of the curve deſcribed by a body acted on by the above mentioned forces: in the fluxional equation found before for D and D′ and their fluxions ſubſtitute $(x^2 + y^2)^{\frac{1}{2}}$ and $((SS' \pm x)^2 + y^2)^{\frac{1}{2}}$ and their fluxions, and there reſults the equation ſought.

Cor. It eaſily appears, that the general fluent may contain two invariable quantities to be aſſumed at will, or according to the conditions of the problem; that is, at a given diſtance the velocity and the direction may be aſſumed at will, and conſequently the general fluxional equation expreſſing the above mentioned relation will be of the ſecond order, if no fluents are contained in it.

Cor. From Py and Py', and the points S and S′ being given, can eaſily be deduced geometrically the direction of the tangent and the lines Sy, Sy', &c.; for divide the line SS′ in r, ſo that $Py \pm Py' : SS' :: Py : Sr$, and through r draw the line Pr, the perpendicular to Pr through P will be the tangent yPy'; to this line the perpendiculars from S and S′ will be the lines Sy and S$'y'$ required.

Cor. From the fluent of the above-mentioned fluxional equation may be deduced the velocity V in terms of D and D′; and from the fluent of $\frac{D\times\dot{D}}{Py\times V}$, which is a function of D multiplied into $\dot{D}$, may be deduced the time.

3. If the plane in which the body (P) moves, and all the forces f', f'', f''', &c. tending to points M′, M″, M‴, &c. not ſituated in the ſame plane (except one f tending to a given point M) be given, then the force tending to that point can be found, and the curve deſcribed. Reſolve all the forces tending to the points M, M′, M″, M‴, &c. into two others; one MS, M′S, M″S, M‴S, &c. perpendicular to the plane in which the body moves, and the other SP, S′P, S″P, S‴P, &c. in the plane; then will $f\times\frac{MS}{MP}\pm f'\times\frac{M'S'}{M'P}\pm f''\times\frac{M''S''}{M''P}\pm\&c.=o$, from which equation f the force tending to the point M may be found; then, from the preceding propoſition find the curve, which a body agitated by forces $f\times\frac{SP}{MP}$, $f'\times\frac{S'P}{M'P}$, $f''\times\frac{S''P}{M''P}$, &c. tending to the points S, S′, S″, &c. deſcribes, and it will be the curve required.

4. If the body moves in a curve of double curvature, and the forces f, f', f'', &c. tending to all the centers M, M′, M″, M‴, &c. be given; from the fluent of the fluxional quantity $(f\times\frac{Py}{MP}\pm f'\times\frac{Py'}{M'P}\pm f''\times\frac{Py''}{M''P}\pm f'''\times\frac{Py'''}{M'''P}\ \&c.)\times\dot{A}$ (A denoting the ſame quantity as before) $=f\times M\dot{P}\pm f'\times M'\dot{P}\pm f''\times M''\dot{P}\pm f'''\times M'''\dot{P},\pm\&c.=f\times\dot{D}\pm f'\times\dot{D}'\pm f''\times\dot{D}''\pm f'''\times\dot{D}'''\pm\&c.=\dot{Z}=-v\dot{v}$ (f, f', f'', f''', &c. being given functions of D, D′, D″, &c. reſpectively) can be deduced the ſquare of the velocity $=-2z$, which will be a function of

D, D', D'', D''', D'''', &c., and conſequently a function of D, D', D'', eaſily to be derived: ſubſtitute this function $-2z$ for v^2 in the two following equations $\frac{v^2}{R'} = F'$ and $\frac{v^2}{R''} = F''$, where R' and R'' denote the radii of curvature in two different planes of which the tangent above mentioned in Prob. 4. art. 4. is their interſection, and F' and F'' the ſum of the forces in lines perpendicular to the tangent, and in the reſpective planes: from theſe forces, calculated in terms of the diſtances from three given points D, D', and D''; or in terms of two abſciſſæ and one ordinate, and from the radii R' and R'' may be deduced two fluxional equations of the ſecond order, expreſſing the relation between three diſtances D, D', and D'', &c. which may always be reduced to one fluxional equation of the fourth order expreſſing the relation between one abſciſs and its correſpondent ordinates, or the diſtances from two given points.

5. The general fluxional equation expreſſing the relation between the diſtances from two given points will be of the fourth order, if no fluents are contained in it; for it admits of four different quantities to be aſſumed at will, or according to the conditions of the problem.

6. If ſome points, to which the forces tend, are ſituated at an infinite diſtance; that is, ſome forces always act parallel to themſelves; from the given forces acting either to given points, or in parallel directions, by the equation $f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm \text{\&c.} = -v\dot{v}$ can be deduced the ſquare of the velocity at a point P in terms of the diſtances from two given points, or of an abſciſs and ordinate; if the centers, &c. and parallel forces are all ſituated in the ſame plane: or in terms of the diſtances from three points, or two abſciſſæ and an ordi-

nate, if ſituated in different planes; from the centers, &c. and forces given, find the ſum F of the forces in any direction (PL) (the direction of the tangent excepted) acting on the body at the point P, and the chord of curvature C of the curve at the ſame point and in the ſame direction; in the equation $v^2 = \frac{1}{2} F \times C$ for v^2 ſubſtitute the value found before, and there reſults an equation expreſſing the relation between the diſtances from two points, or an abſciſs and ordinate, &c. if the forces act in the ſame plane: but if the forces act in different planes, find the ſum F and F′ of the forces at the point P in directions which are not both ſituated in one plane with the tangent and each other; and alſo the chords C and C′ of curvature in thoſe directions in terms of the diſtances from three points, or two abſciſſæ and one ordinate, &c. In the equations $v^2 = \frac{1}{2} F \times C$ and $v^2 = \frac{1}{2} F' \times C'$ for v^2 ſubſtitute its value found from the principles before given; and there reſult two fluxional equations of the ſecond order expreſſing the relation between the diſtances from three points, or two abſciſſæ and an ordinate, &c.

PROP. VIII.

Fig. 9. Let a body move in a curve P*p*, &c. and be acted on at P′ by a force *f* (which is as any function of the diſtance SP′) tending to S; let the velocities at P and *p* be repreſented by the lines YP and *yp* in the direction of the tangents to the points P and *p*; reſolve theſe forces YP and *yp* into two others Y*k* and *k*P, and *yl* and *lp*, of which one *k*Y and *yl* is parallel to the line SL; the other *k*P and *lp* is parallel to MP: let a body fall in the right line LS, and the force acting on the body at M′ be to the force acting on the body moving in the curve at P′ :: SM′ : SP′, and P′M′, PM

and pm be perpendicular to SL; then, if the velocity of the body falling in the right line SL at the point M be kY, the velocity of the body at the point m acted on by the above mentioned forces will be yl.

This is eaſily demonſtrated from the reſolution of forces.

2. Through S draw SN parallel to PM or pm, &c., and aſſume in the line (SN) $SP = PM$ and $Sp = pm$, and let the force at P' in the line SN and diſtance $= M'P'$: the force of the body moving in the curve at the diſtance $P'S : P'M' : SP'$; then if the velocity at the diſtance $SP = PM$ be Pk, the velocity at the diſtance $Sp = pm$ will be pl.

Cor. The force in the direction of the line SL vaniſhes in the point where a perpendicular SN to the line SL paſſing through the point S cuts the curve, and conſequently the velocity in the direction of SL in that point is the greateſt or leaſt, &c.; but if the tangent of the curve be perpendicular in any point to LS, then the velocity in the direction LS is nothing: the ſame may be applied to the velocity in any other direction.

Ex. Fig. 10. Let a body move in the circumference of a circle SPA, of which the center of force is a point S in the circumference; it is known, that the force in the direction and at the diſtance SP is as SP^{-5}; but the force in the direction SP is by the hypotheſis to the force in the direction (SA) :: SP : SM, if PM be perpendicular to SM, and conſequently the force in the direction (SA) is as $SM \times SP^{-6}$; but if AS be a diameter, $AS \times SM = SP^2$; therefore $SM \times SP^{-6} = SM \times AS^{-3} \times SM^{-3} = \frac{SM^{-2}}{AS^3}$; and the diameter AS being given, the force in the line SA varies as SM^{-2}, that is, inverſely as the ſquare of the diſtance: if the force varies as $SM^{-2} = x^{-2}$, then $v\dot{v}$ will vary as

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$-\frac{\dot{x}}{x^2}$, where v denotes the velocity; and v^2 will vary as $\frac{1}{x}-\frac{1}{SA}$, which agrees with the ſquare of the velocity deduced from the preceding principles; for $v=PY$ the velocity at P is inverſely as the perpendicular $SY=SM$ let fall from the center of force on the tangent; but $SA^2 : 2SP \times PA ::$ velocity PY as $\frac{1}{SY}=\frac{1}{SM}$: Pl the velocity at M; whence Pl^2 (the ſquare of the velocity at M) $=\frac{4SP^2 \times PA^2}{SA^4} \times PY^2$ which varies as $\frac{4SP^2 \times PA^2}{SA^4} \times \frac{1}{SM^2}=\frac{4PA^2}{SA^3 \times SM}=\frac{4SA^2-4SA \times x}{SA^3 \times x}$, and conſequently as $\frac{1}{x}-\frac{1}{SA}$ the ſame as above.

2. Fig. 9. If any number of forces act on a body at P in any given directions parallel, or tending to given points; reſolve all the forces into two others; one in a given direction SM, and the other in a direction PM perpendicular to it, of which let F be the ſum of the forces reſulting in the direction M*m*S, and *f* the ſum of the forces reſulting in the direction PM; reſolve the velocity V of the body at P, which is in the the direction of the tangent PY, into two others V' and V'', one in the direction parallel to the line SM, and the other perpendicular to it: in the ſame manner reſolve the velocity v of the body at p, which is in the direction of the tangent py, into two others v' and v'', one in the direction parallel to the line SM, and the other perpendicular to it: then if the velocity of the body moving in the right line SM at M be V', and it is conſtantly acted on by a force $=F$, the velocity of the body at m will be v': and if the body move from P in a direction perpendicular to SM with a velocity as V'', and be always acted on by a force f, the velocity at the diſtance $PM-pm$ will be v''.

Cor. From the forces given and the velocities in the above mentioned directions at the point P, can be deduced the velocities in the ſame directions at the point p, and conſequently the tangent to the curve at the point p.

PROP. IX.

1. Let the reſiſtance of a body, moving in a right line, be as any function V of the velocity v; then will $\dot{t} = \frac{\dot{v}}{V}$, $\dot{x} = \frac{-v\dot{v}}{V}$; where $\dot{t}$, $\dot{v}$, and $\dot{x}$, denote the increments of time, velocity, and ſpace; their fluents properly corrected will give the time and ſpace in terms of the velocity.

2. Let a body move in a right line, and be acted on by an accelerating force in that line, which varies as any function X of the diſtance x from a given point; and reſiſted by a force which is as any function V of the velocity v into its denſity X′, which varies alſo as a function of x and v; then will $(X + aVX')\dot{x} = -v\dot{v}$, from its fluent x can be found in terms of v, or v in terms of x; and thence $\dot{t} = \frac{\dot{v}}{X + aVX'}$, of which the fluent properly corrected gives the time.

Ex. 1. Let $V = v^2$ and X′ a function of x; that is, let the reſiſtance be as the ſquare of the velocity and denſity, whence $(X + av^2X')\dot{x} = -v\dot{v}$, of which equation the fluential will be

$$e^{\int 2aX'\dot{x}} \times \frac{v^2}{2} = -\int e^{\int 2aX'\dot{x}} \times X\dot{x} + A, \text{ and } t =$$

$\int \frac{\dot{x}}{\sqrt{-(e^{-\int 2aX'\dot{x}} \times (\int e^{\int 2aX'\dot{x}} \times X\dot{x} + A))}} + B$, where A and B are invariable quantities to be aſſumed according to the conditions of the problem.

1.2.

1.2. Let $e^x = X'$ and $X = b$, which is ſuppoſed to correſpond nearly to the ſtate of our atmoſphere, then will $v^2 = -2 \times e^{-\int 2aX'\dot{x}} \times \int e^{\int 2aX'\dot{x}} \times X\dot{x} = -2 \times e^{-\int 2ae^x\dot{x}} \int e^{\int 2ae^x\dot{x}} \times b\dot{x} = -2e^{-2ae^x - b} \times \left(\int e^{2ae^x + b} \times b\dot{x} + A\right)$, e being the number, whoſe hyperbolic log. is 1, and b and A quantities to be aſſumed according to the conditions of the problem.

1.3. Let $X = X'$, and it becomes $X\dot{x} = \frac{-v\dot{v}}{1 + aV}$, and $\dot{t} = \frac{-\dot{v}}{X(1 + aV)}$.

2. Let X be an homogeneous function of one dimenſion of x, that is, $= ax$, and V a ſimilar function of n dimenſions of v, that is $= bv^n$, and X' a ſimilar function of r dimenſions of x and v, and $n + r = 1$; then by ſubſtituting zx and its fluxion for v and its fluxion, can be found the fluent of the fluxional equation $(X + aVX')\dot{x} = -v\dot{v}$, and conſequently the velocity and time by the quadrature of curves in terms of the ſpace; and in like manner of many other caſes.

3. Fig. 4. Let a body moving in a given curve be acted on at any point P by a force f tending to a given point S, and reſiſted by a medium proportional to V a function of its velocity multiplied into its denſity X' a function of the diſtance $SP = D$; to find its velocity, time, and diſtance from the given point S in terms of each other. Let $F = f \times \frac{PY}{SP}$ the force in the direction of the tangent PY, and conſequently $(F + VX')\dot{A} = -v\dot{v}$, and $v^2 = \frac{1}{2} C \times f$, where $\dot{A}$ is the increment of the arc, and C the chord of curvature in the direction SP; but ſince the curve is given, the chord of curvature may be deduced from the diſtance, &c. and the increment $\dot{A}$ of the arc from a function of the diſtance multiplied into the increment

of the diſtance; then, if f or v be a given function of the diſtance, the other may be deduced from it, and conſequently $-v\dot{v}=\varphi:(\mathrm{D})\times\dot{\mathrm{D}}$ will be a given function of the diſtance D multiplied into $\dot{\mathrm{D}}$, whence we have $\varphi:(\mathrm{D})\times\dot{\mathrm{D}}=\dot{\mathrm{D}}\,(f\times\frac{\mathrm{PY}}{\mathrm{SP}}+\mathrm{X'V})$ divide by $\dot{\mathrm{D}}$, and there reſults an algebraical equation, from which $\mathrm{V}\times\mathrm{X'}$ may be found.

If neither v nor f be given, reduce the two equations $(f\times\frac{\mathrm{PY}}{\mathrm{SP}}+\mathrm{VX'})\,\dot{\mathrm{A}}=-v\dot{v}$ and $v^2=\frac{1}{2}\mathrm{C}f$ into one, ſo as to exterminate either f or v and its fluxions, and there reſults an equation expreſſing the relation between the other v or f and D and their fluxions: from the velocity given in terms of D may be deduced the time from the equation $\dot{t}=\frac{\dot{\mathrm{A}}}{v}$.

3.2. If the body be acted on by forces tending to more points S, S′, S″, S‴, &c. in the ſame plane; reſolve each of the forces into two; one in the direction of the tangent, and the other perpendicular to it; let the ſum of the forces in the direction of the tangent be F; and in the direction perpendicular to it be F′; and 2R the diameter of curvature at the point P, which will be given in terms of the diſtances from two points, or of an abſciſs and ordinate, and their fluxions, &c.: aſſume the two equations before given $(\mathrm{F}+\mathrm{X'V})\,\dot{\mathrm{A}}=-v\dot{v}$ and $v^2=\mathrm{F'R}$, and ſince $\dot{\mathrm{A}}$ is always given in terms of D and $\dot{\mathrm{D}}$, if F and F′ be given in terms of D, D′, &c. the value of $\mathrm{V}\times\mathrm{X'}$ may be acquired by a ſimple algebraical equation: but if F and F′ be not given, and conſequently v not given, but V a given function of v, and X′ a given function of the above mentioned diſtances; then ſubſtitute for v its value $\sqrt{(\mathrm{F'R})}$ in the function V, and the fluxion of $\frac{1}{2}\,\mathrm{F'R}$ for $v\dot{v}$, and there will reſult an equation involving D and F′ and their fluxions, and F; but if the forces tending to all the points but one are given in terms of the diſtance D, or abſciſs or ordinate of the curve, and their fluxions; then from F′ can be found F, and, *vice verſâ*, from F can be found F′, and conſequently there reſults a fluxional equation expreſſing the relation between F or F′ and the diſtance D or D′, &c. or abſciſs or ordinate, and their fluxions.

From F and F′, and conſequently v being found in terms of D, D′, &c. can be deduced $\dot{t} = \frac{\dot{v}}{F}$.

The ſame method may be applied, if ſome forces tend to an infinite diſtance, that is, act parallel to themſelves, and others tend to given points.

Ex. Let the accelerating force be directly as the arc $= x$, and the reſiſtance uniform $= a$; then will $(x - a)\,\dot{x} = -v\dot{v}$, and conſequently $x^2 - 2ax + B = -v^2$; let A be the arc, where the velocity $= o$; then will the equation $A^2 - 2aA - x^2 + 2ax = v^2$, and the increment of the time $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{(A^2 - 2aA - x^2 + 2ax)}}$, whoſe integral is $\frac{1}{A-a} \times$ arc of a circle, of which the radius is $A - a$ and coſ. $= x - a$, where A is the diſtance of the point from which the body begins to fall, and the loweſt point of the curve; and the accelerating force $x - a$ is as the diſtance from a point (a) of a curve, of which the diſtance from the loweſt is a.

Cor. The times of the body falling from any point of the curve to a will be equal.

Cor. The body on this hypotheſis will either reſt at the point a, or at the loweſt point, or any point between $+a$ and

$-a$; for it may reſt at any point, where the reſiſting force is always equal or greater than the accelerating force.

Cor. Let n be the number of vibrations, then the diſtance of the arc, to which it will aſcend from the loweſt point at n vibrations, will be $A-2na$; if $A-2na$ be not greater than $2a$, it will never paſs the loweſt point.

Philoſophical enquiries require ſome corrections, which do not enter into mathematical calculus; for example, in ſome caſes the calculus changes the quantities from negative to affirmative, &c. when from philoſophical conſiderations they are not changed; and, *vice verſâ*, they may be changed to affirmative, &c. on philoſophical conſiderations, when they are not changed from the calculus: and alſo a body may ſtop, &c. from philoſophical conſiderations, as in the preceding example, when it does not follow from the algebraical calculus, &c. It is further to be obſerved, that reſiſtances are always to be taken affirmatively.

Ex. 2. Let the accelerating force be as the arc, that is, the diſtance from the loweſt point, and the reſiſtance as the velocity; then will the fluxional equation $(F-V)\,\dot{A}=-v\dot{v}$ be $(ax-v)\,\dot{x}=-v\dot{v}$, which is an homogeneous equation of the firſt order: write in it zx for v, and its fluxion for $\dot{v}$, and there reſults the equation $(ax-zx)\times\dot{x}=-zx^2\dot{z}-z^2x\dot{x}$, whence $(a-z)\,\dot{x}=-zx\dot{z}-z^2\dot{x}$ and $\frac{\dot{x}}{x}=\frac{-z\dot{z}}{a-z+z^2}$, and thence log. $x=-\frac{1}{2}$ log. $(a-z+z^2)$ (W) $-\frac{2}{4a-1}\times$ cir. arc, whoſe radius is $\frac{\sqrt{4a-1}}{2}$ and tangent $(z-\frac{1}{2})+B$, whence can be found $v=xz$, and from curvilinear areas $\dot{t}=\frac{\dot{x}}{v}$.

If $4a$ is leſs than 1, then it becomes log. $x = \mathrm{W} - \frac{1}{4\sqrt{\frac{1}{4} - a}} \times$ log. $\frac{z - \sqrt{\frac{1}{4} - a} - \frac{1}{2}}{z + \sqrt{\frac{1}{4} - a} - \frac{1}{2}} + \mathrm{B}$; where B is an invariable quantity to be aſſumed according to the conditions of the problem.

Cor. If the force be directly as the diſtance, or as the arc of the curve from the body to the loweſt point, and the reſiſtance as the velocity; then will the velocity in one arc be to the velocity in the correſponding point of another arc, as the arcs to be deſcribed; and conſequently the times equal.

4. If the body is acted on by forces tending to points S, S′, S″, &c. ſituated in different planes, then let F be the ſum of the forces in the direction of the tangent at the point P; F′ and F″ the ſum of the forces acting on the body in two different directions at the ſame point, which are not ſituated in the ſame plane with the tangent and each other; from the three equations $(\mathrm{F} + \mathrm{X'V})\,\dot{\mathrm{A}} = -v\dot{v}$ and $\frac{v^2}{\mathrm{C}} = \frac{1}{2}\mathrm{F}'$ and $\frac{v^2}{\mathrm{C}} = \frac{1}{2}\mathrm{F}''$, in which the ſame letters denote the ſame quantities as before, and C and C′ denotes the chords of curvature in the ſame directions as the forces F′ and F″, which from the curve being given can be found at any point; and if F′ or F″ is given in terms of the diſtance from a given point, or an abſciſs or ordinate, &c. the velocity v can be found in terms of the ſame, and X′V by a ſimple algebraical equation: if F′ is not given, and V is a given function of v, ſubſtitute in V for v its value $\sqrt{(\frac{1}{2}\mathrm{C} \times \mathrm{F}')}$, and there reſults an equation expreſſing the relation between F (which can be deduced from F′ or F″) and the diſtance of the body from ſome given point, or the abſciſſæ and ordinates of the curve required, and their fluxions.

If ſome of the forces act in parallel directions; the forces, velocities, &c. may be found by the ſame method.

PROP. X.

Fig. 11. Let a body be projected in a direction HL with a given velocity, and be acted on by a force in a direction parallel to AP $= x$, which varies as X a function of x; and alſo by another force in a direction parallel to MP $= y$, that is, perpendicular to AP, which force varies as Y a function of y; and let it move in a medium, of which the reſiſtance is proportional to the velocity; to find the curve deſcribed.

Find the fluent of $(X + av)\,\dot{x} = -v\dot{v}$, which corrected according to the conditions of the problem (*viz.* ſo that v at the point H may be to the velocity of projection :: Hc : Hb, where bc is drawn perpendicular to AP) ſuppoſe $v = X'$; find the fluent of $\frac{\dot{x}}{X'}$, which corrected ſo as to become $= o$, when $x = AH$, let be X''. In the ſame manner find the fluent of $(Y + a'v')\,\dot{y} = -v'\dot{v}'$, which corrected, ſo that v' at the point H may be to the velocity of projection :: cb : Hb, ſuppoſe $v' = Y'$; find the fluent of $\frac{\dot{y}}{Y'}$, which corrected ſo as to become $= o$, when $PM = o$, let be $= Y''$; aſſume $X'' = Y''$, and thence from x find y: take $AP = x$ and $PM = y$, and M will be a point in the curve, which a body projected in the line HL deſcribes; and if Mm in the direction parallel to HAP : mo perpendicular to it :: velocity v : velocity v', then will Mo be a tangent to the curve in the point M.

2. If a body is acted on by forces tending to any given points S, S′, S″, &c. which vary as given functions of their diſtances from the body, and reſiſted by a force which varies according to a given function V of the velocity (v) into its denſity X′,

where X' varies according to ſome function of the diſtances from the given points, &c.; to find the curve deſcribed.

1. From the diſtances of the body from two given points, or the abſciſs and ordinate of the curve deſcribed, and their fluxions, &c. find the forces acting in the direction of the tangent to the curve, and in ſome other direction, which ſuppoſe F and F'; and alſo the chord of curvature in the above mentioned direction, which let be C; then from the equations $(F+V\times X')\dot{A} = -v\dot{v}$ and $v^2 = \frac{1}{2}C\times F$ reduced into one by writing for v its value in the function V, and for $v\dot{v}$ its value deduced from the equation $v^2 = \frac{1}{2}C\times F$, and for $\dot{A}$ (the fluxion of the arc) its value deduced from the diſtances, &c. will reſult an equation expreſſing the relation between the diſtances from two given points to the curve, or its abſciſs and ordinates, and their fluxions.

3. If the forces are not all ſituated in the ſame plane, then from the before given equation $(F+V\times X')\dot{A} = -v\dot{v}$, and the two others $v^2 = \frac{1}{2}C\times F'$ and $v^2 = \frac{1}{2}C'F''$, where F denotes the force in the direction of the tangent, and F' and F'' are the forces in different directions, which both are not ſituated in the ſame plane with each other and the tangent, and in which directions the chords of curvature are reſpectively C and C'; ſince the quantities F, F', and F''; C and C' and $\dot{A}$ (as proved before) can all be expreſſed in terms of the diſtances from three given points, or from two abſciſſæ and one ordinate, and their reſpective fluxions; may be deduced two fluxional equations expreſſing the relation between the diſtances from three given points, or two abſciſſæ and an ordinate, &c.

The ſame principles may be applied to caſes, in which ſome of the forces act in parallel directions.

On moveable Centers.

PROP. XI.

1. Given the reſpective places of (n) bodies S, S', S'', S''', &c. in the curves A, A', A'', A''', &c. at the ſame time, and in the ſame plane, and the forces of all the bodies acting on S, except two, S' and S''; to find the forces of the two bodies S' and S'' on the body S.

This propoſition may be reſolved by the method given in Prop. 4. for to produce the ſame effect the ſame finite forces will be requiſite, whether the centers of forces reſt or move in given curves.

1.2. If the bodies S, S', S'', &c. move in different planes, then all the forces acting on the body, except three, may be given, which may be acquired from the method given in the ſame propoſition.

Hence it appears, that $2n$ forces may be requiſite to be found from the conditions of the problem to determine all the bodies to move in their reſpective curves, when they are all ſituated in the ſame plane, and that $3 \times n$ forces may be requiſite in different planes, &c. if the force of one body (S') on another (S'') does not at all depend on the force of the ſame body (S') on any other (S'''); and if the ſame can be prædicated of the reſt, then $n \cdot \overline{n-3}$ forces of the above mentioned bodies in the ſame, or $n \cdot \overline{n-4}$ forces in different planes may be aſſumed at will.

3. If the velocities v, v', v'', &c. at every point of the arcs a, a', a'', &c. of the (n) above mentioned curves A, A′, A″, &c. be given in terms of their arcs, abſciſſæ, or ordinates, &c. and the places in which the bodies are ſituated at the ſame time in the arcs b, b', b'', &c. of ſome other curves B, B′, B″, &c. find the correſponding velocities V, V′, V″, &c. at the ſame time of the bodies in the curves B, B′, B″, &c.; then make $\frac{\dot{a}}{v} = \frac{\dot{a}'}{v'} = \frac{\dot{a}''}{v''} = \&c. = \frac{\dot{b}}{V}$, or which is equal to it $= \frac{\dot{b}'}{V'}$ or $= \frac{\dot{b}''}{V''} = \&c.$ From the fluents of the fluxional equations reſulting properly corrected will be found the arcs a, a', a'', &c. deſcribed by the bodies in the curves A, A′, A″, &c. in the ſame time as the correſpondent arcs b, b', b'' &c.; and from thence, by the method given in the preceding caſe, may be deduced the forces.

The ſame principles may be applied to bodies moving in reſiſting mediums.

PROP. XII.

Given the law of the forces of two bodies acting on each other, to find the two curves by them deſcribed.

Fig. 12. Aſſume x and y for the abſciſs (AP) and ordinate (PM) of one curve, and z and u for the abſciſs (AP′) and ordinate (P′M′) of the other; where the abſciſſæ AP and AP′ begin from the ſame point A, and are ſituated in the ſame line; then will the diſtance (D = M′M) between the bodies $= \surd\overline{z \pm x}^2 + \surd\overline{u \pm y}^2)$; let the forces of the body placed at M on that at M′, and of the body placed at M′ on that at M vary as $\varphi : (D) = F$, and $\varphi' : (D) = F'$; and let $M\dot{p} = \dot{x}$ and $\dot{p}m = \dot{y}$; then will coſine of the angle mMM′ to radius (1) be $\frac{y \pm u}{D} \times$

$\dot{y}$

$\frac{\dot{y}}{\surd(\dot{x}^2+\dot{y}^2)} \pm \frac{x \pm z}{D} \times \frac{\dot{x}}{\surd(\dot{x}^2+\dot{y}^2)} = c$; and conſequently the force in the direction of the tangent M*m* will be $c \times F$, whence $-v\dot{v} = c \times F \times \sqrt{\dot{x}^2+\dot{y}^2}\ (\dot{A})$ and $v^2 = \frac{1}{2}CF$, where C is the chord of curvature in the direction of the force $(F) = \sqrt{1-c^2} \times 2\frac{(\dot{x}^2+\dot{y}^2)^{1\frac{1}{2}}}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$; and v the velocity of the body in the curve, whoſe abſciſs is x and ordinate y.

In the ſame manner let $\frac{x \pm z}{D} \times \frac{\dot{z}}{\surd\dot{z}^2+\dot{u}^2} \pm \frac{y \pm u}{D} \times \frac{\dot{u}}{\surd\dot{z}^2+\dot{u}^2} = c'$, the coſine of the angle made between the diſtance MM′ and arc of the curve of which the abſciſs is z and ordinate u, and conſequently $c' \times F'$ will be the force in the direction of its tangent, and therefore $-v'\dot{v}' = c' \times F' \times \sqrt{\dot{z}^2+\dot{u}^2}\ (\dot{A})$ and $v'^2 = \frac{1}{2}C'F'$, where C′ is the chord of curvature in the direction of the force $(F') = \sqrt{1-c'^2} \times 2\frac{(\dot{z}^2+\dot{u}^2)^{1\frac{1}{2}}}{(\dot{u}\ddot{z}-\dot{z}\ddot{u})}$, and v' the velocity of the body in the curve whoſe abſciſs is z and ordinate u; then, becauſe the times of deſcribing correſpondent arcs in the two curves are equal, their increments will be equal, and conſequently $\dot{t} = \frac{\surd\dot{x}^2+\dot{y}^2}{v} = \frac{\surd\dot{z}^2+\dot{u}^2}{v'}$; and there are deduced five fluxional equations, containing ſix variable quantities v, v'; x, y; z, and u, and their fluxions; reduce theſe equations, ſo that four of them (v, v', &c.) may be exterminated, and there will reſult an equation expreſſing the relation between x and y the abſciſs and ordinate of one curve, or z and u the abſciſs and ordinate of the other curve, and their fluxions; the fluential equation of which being found, and properly corrected, gives the equation to the curve.

The five equations are eaſily reduced to three by extermінating the quantities v and v'.

The fluxional equation reſulting will moſt commonly be of the fifth order, as evidently appears from the nature of the problem.

2. The ſame principles may be applied to determine the curves, when the bodies move in mediums, of which the reſiſtances are given: for example, ſuppoſe the reſiſtances to vary as a function of the diſtance from a given point into a function of the velocity: to the forces in the directions of the tangents contained in the preceding caſe muſt be added or ſubtracted the given reſiſtances for the forces in the directions of the tangents, and the remaining proceſs will be the ſame as is before given.

If two bodies deſcribe ſimilar orbits round a common center, either quieſcent or moving uniformly in a right line; the forces and velocities and reſiſtances of the medium will be to each other in correſpondent points as their reſpective diſtances from the center.

PROP. XIII.

Given the forces acting on any bodies, and tending to points either moveable or quieſcent, or in the direction of the tangents, &c.; to find the curve deſcribed by one of the bodies.

1. Aſſume x and y for the abſciſs and ordinate of the curve required, and from thence may be deduced the diſtances from any quieſcent center of force, and conſequently the force f in that direction; reſolve it into two others, one in the direction of the tangent, and the other in a different one; for example, let it be in a direction perpendicular to the tangent, and from their fluxions $\dot{x}$ and $\dot{y}$, and the force f may, by the method

before given, be deduced the forces in the two above mentioned directions; and in the ſame manner may be found from x, y, $\dot{x}$, and $\dot{y}$, the forces in the directions of the tangent and perpendicular to it, which follow from all the forces tending to given points, and acting on the body moving in the curve to be inveſtigated. 2. If ſome of the centers of force move in given curves B, B$'$, B$''$, &c. whoſe arcs let be denoted by B, B$'$, &c. and their reſpective places at the ſame time are given; then from their reſpective places given and forces, and x and y, and $\dot{x}$ and $\dot{y}$, can, as before, be deduced the forces in the direction of the tangent and its perpendicular to the curve required. 3. If other centers of forces move in given curves A, A$'$, A$''$, &c. and the velocities are given at every point of the curves; let A, A$'$, A$''$, &c. be the arcs of the curves A, A$'$, A$''$, &c. and ſuppoſe v, v', v'', &c. their correſpondent velocities; then, if the increments of the time be given, will $\frac{\dot{A}}{v} = \frac{\dot{A}'}{v'} = \frac{\dot{A}''}{v''} =$ &c. but as the velocities are given at every point of the curves, v in the curve (A) will be given in terms of its abſciſs, ordinate, arc, &c. and conſequently $\frac{\dot{A}}{v}$ in terms of the ſame quantities and their firſt fluxions; the ſame may be affirmed of the fluxions $\frac{\dot{A}'}{v'}$, $\frac{\dot{A}''}{v''}$, in the curves A$'$, A$''$, &c.; hence, from the equation $\frac{\dot{A}}{v} = \frac{\dot{A}'}{v'}$, can be deduced the relation between the abſciſs or ordinate, &c. of the curve A and its correſpondent abſciſs or ordinate, &c. of the curve A$'$; and ſo of the remaining curves; hence this caſe is reduced to the preceding; but it is neceſſary alſo, that the times of the bodies in the two caſes ſhould be the ſame, in order that the places may correſpond, and conſequently $\frac{\dot{A}}{v} = \frac{\dot{B}}{V}$, where V denotes the velocity of the body at any point of the curve B from which equation can be deduced the correſpondent abſciſſæ and ordinates, &c. of the curves B and A; and thence the two caſes are reduced to the preceding, whence the correſpondent forces in the directions of the tangent, and perpendicular to it, can be found as above. 4. If ſome (m) of the centers move in curves L, L′, L″, &c to be deduced from the laws of the forces being given which act on them; aſſume z and u, z' and u', z'' and u'', &c. for their reſpective abſciſſæ and correſpondent ordinates; and from them and y and x, $\dot{y}$ and $\dot{x}$, find the forces acting on the body moving in the curve required in the direction of the tangent, and perpendicular to it, as before; then add all the forces deduced which act perpendicular to the tangent and alſo all contained in the direction of the tangent together with the reſiſting force in the ſame direction, and let the ſums reſulting be reſpectively F and F′: by the ſame method find the ſum of the forces which act on the bodies moving in L, L′, L″, &c. in the directions of the tangents, and perpendiculars to them, which ſuppoſe S and s, S′ and s', S″ and s'', &c.; then reduce the $2\,(m+1)$ equations of the formulæ found above, *viz.* $v^2 = F \times \frac{\overline{\dot{x}^2+\dot{y}^2}^{\,1\frac{1}{2}}}{\ddot{y}\dot{x}-\ddot{x}\dot{y}}$ and $-v\dot{v} = F' \sqrt{\dot{x}^2+\dot{y}^2}$; $v'^2 = s \times \frac{\overline{\dot{z}^2+\dot{u}^2}^{\,1\frac{1}{2}}}{\dot{u}\ddot{z}-\dot{z}\ddot{u}}$ and $-v'\dot{v}' = S \times \sqrt{\dot{z}^2+\dot{u}^2}$; $v''^2 = s' \times \frac{\overline{\dot{z}'^2+\dot{u}^2}^{\,1\frac{1}{2}}}{\dot{u}'\ddot{z}'-\dot{z}'\ddot{u}'}$ and $-v'' \times \dot{v}'' = S' \times \sqrt{\dot{z}''^2+\dot{u}''^2}$, &c.; where v, v', v'', v''', &c. reſpectively denote the correſpondent velocities of the bodies moving in the curves, whoſe abſciſſæ are x, z, z', z'', &c.; and alſo the $(m+1)$ equations $\frac{\dot{B}}{V} = \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v} = \frac{\sqrt{\dot{z}^2+\dot{u}^2}}{v'} = \frac{\sqrt{(\dot{z}'^2+\dot{u}'^2)}}{v''} = \frac{\sqrt{(\dot{z}''^2+\dot{u}''^2)}}{v'''} =$ &c. containing the $3\,(m+1')$

$+1$ variable quantities x and y, z and u, z' and u', z'' and u'', &c., v, v', v'', &c., and the variable quantity contained in B and V, into one, ſo that all the variable quantities except x and y and their fluxions may be exterminated, and there reſults an equation to the curve required expreſſing the relation between x and y its abſciſs and ordinate, and their fluxions.

5. If the forces are not ſituated in the ſame plane, aſſume X, x and y, for the two abſciſſæ and ordinates of the curve required; and Z, z and u; Z′, z' and u'; Z″, z'' and u''; &c. for the two abſciſſæ and ordinates of the (m) curves L, L′, L″, &c. reſpectively; and from the preceding method may be acquired the $3(m+1)$ equations $v^2 = F \times C$, $v^2 = F' \times C'$, and $-v\dot{v} = F'' \times \sqrt{\dot{X}^2 + \dot{x}^2 + \dot{y}^2}$; $v'^2 = S \times C'$, $v'^2 = \sigma c'$ and $-v'\dot{v}' = s \times \sqrt{Z^2 + \dot{z}^2 + \dot{u}^2}$; $v''^2 = S'C'' = \sigma' c''$ and $-v''\dot{v}'' = s' \times \sqrt{(\dot{Z}'^2 + \dot{z}'^2 + \dot{u}'^2)}$; $v'''^2 = S''C''' = \sigma'' c'''$ and $-v'''\dot{v}''' = s'' \times \sqrt{(\dot{Z}''^2 + \dot{z}''^2 + \dot{u}''^2)}$; &c.; in which v denotes the velocity in the required curve, and v', v'', v''', &c. the correſpondent velocities in the curves L, L′, L″, &c.; and F, F′, and F″; S, σ and s; S′, σ' and s'; S″, σ'' and s''; &c. denote the forces acting on the reſpective bodies in two different planes and in the tangents, which planes cut each other in the tangents of the curves; and C and c, &c., C′ and c', &c., C″ and c'', &c. the $\frac{1}{2}$ chords or radii of curvature in thoſe two planes to the different curves in the directions of the forces; and alſo the $(m+1)$ equations beforementioned $\frac{\dot{B}}{V} = \frac{\sqrt{\dot{X}^2 + \dot{x}^2 + \dot{y}^2}}{v} = \frac{\sqrt{\dot{Z}^2 + \dot{z}^2 + \dot{u}^2}}{v'} = \frac{\sqrt{Z'^2 + \dot{z}'^2 + \dot{u}'^2}}{v''} = \&c.$; where $\sqrt{\dot{X}^2 + \dot{x}^2 + \dot{y}^2}$, $\sqrt{\dot{Z}'^2 + \dot{z}'^2 + \dot{u}'^2}$, &c. are the fluxions of the arcs of the required curve, and of the curves L, L′, L″, &c. reduce theſe $4m+4$ equations con-

taining $4m+5$ variable quantities into two, ſo that all the variable quantities except three, X, x, and y, and their fluxions may be exterminated; and there reſult the two equations required.

It may be obſerved, that when the reſiſtance ariſing from the denſity of the medium and velocity (v) of the body varies as $X' \times v^2 + X$, where X and X′ are as functions of the diſtances from given points, the reſolution of the fluxional equations will generally be more eaſy, than when the reſiſtance varies as other functions of the velocities.

If the force acts equally on the particles of the body and fluid, then the force by which a body deſcends in a medium is as the whole force X acting on the body at the given diſtance multiplied into a fraction whoſe numerator is the difference between the denſity of the body (D) and fluid (X′) at that diſtance and denominator D, that is, as $X \times \frac{D - X'}{D}$.

Many caſes might have been given, in which the fluxional equations could have been reſolved; but in general their fluents can only be found by means of converging ſerieſes.